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DIFFERENTIABLE FUNCTIONS ON BANACH SPACES WITH LIPSCHITZ DERIVATIVES

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Introduction

In this paper we study those functions in $C^k(E, F)$, (i.e., functions from two Banach spaces E to F having k continuous Frechet derivatives), whose k-th derivative is Lipschitz with constant M. On R^n we construct C^1 functions whose derivatives are piecewise linear with Lipschitz constant M. From this we obtain a Whitney type extension theorem for real-valued differentiable functions on Hilbert space, and show that every Hilbert space has C^1 partitions of unity. We examine the existence of "nontrivial" C^k functions with Lipschitz derivatives on separable Banach space and show that c_0 has no "nontrivial" C^1 function with Lipschitz derivative. We show that the Whitney extension theorem fails for separable Hilbert space by exhibiting a C^3 function on a closed subset of l^2 having no C^3 extension.

We make the definitions:

$$B_M^k(E, F) = \{f \mid f \in C^k(E, F) \text{ and } \|D^kf(y) - D^kf(x)\| \leq M\|x - y\| \text{ for all } x, y\}\,,$$
$$B^k(E, F) = \{f \mid f \in B_M^k(E, F) \text{ for some } M\}\,.$$

As in Bonic and Frampton [2] a Banach space E is said to be B^k smooth if there is a function $f \in B^k(E, R)$ with $f(0) \neq 0$ and support (f) bounded. Then B^{k+1} smoothness implies B^k smoothness, and E is said to be B^∞ smooth if E is B^k smooth for all k. We briefly summarize some results concerning C^k smoothness of separable Banach spaces. We refer to [2] and Eells [5] for more details.

1. Hilbert space is C^∞ smooth with C^∞ norm away from zero.
2. c_0 is C^∞ smooth with equivalent C^∞ norm away from zero. Kuiper.
3. A Lebesgue space $\mathscr{L}^p$ is C^∞ smooth for an even integer p, and C^{p-1} smooth but not D^p smooth for an odd integer p; Bonic and Frampton [2].
4. If E is separable, then E has a norm in $C^1(E - \{0\}, R)$ if and only if E^* is separable; Bonic and Reis [3].
5. Any C^k smooth separable Banach space has C^k partitions of unity; Bonic and Frampton [2].

In § 2 we prove some basic properties of $B_M^k(E, F)$, the most useful one being that $\{f \mid \|f\| \leq b$ on some open subset of $E\} \cap B_M^k(E, F)$ is closed in the

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topology of pointwise convergence. We observe from [2] that an $\mathscr{L}^p$ space is B^∞ smooth for an even integer p and $B^{[p-1]}$ smooth when p is not. We show that c_0 is not B^1 smooth and that every B^k smooth separable Banach space has B^k partitions of unity. These last two results were announced in Wells [10].

The distance function from a convex set is studied in § 3, and we show that if $\|x\|^2 \in B^1_M(E,R)$ then distance $^2(x,A) \in B^1_M(E,R)$ for closed and convex A.

In § 4 we make a cellular decomposition of R^n on which a B^1_M function is constructed with prescribed values and derivatives at a finite number of points. Using these functions we obtain a necessary and sufficient condition for a real-valued function defined on a closed subset of Hilbert space to have a B^1_M extension to all of Hilbert space. One of the properties of this extension implies that every closed subset of Hilbert space is the zero set of a $B^1(H,R)$ function. Thus a nonseparable Hilbert space has C^1 partitions of unity by an easy construction; this result was announced in Wells [11].

In § 5 we exhibit a closed convex subset in l^2 for which there exists no B^2 function satisfying $f(A) = 0$ and $f(\{x \mid \|d(x,A)\| \geq 1\}) \geq 1$. A corollary of this is that the Whitney extension theorem fails for C^3 functions on Hilbert space. We end the section with some open problems.

2. B^k functions and B^k smooth Banach spaces

If f has a j-th Frechet derivative at x, we will let $D^jf(x)[h]$ denote the j-multilinear map $D^jf(x)$ acting on $(h, \cdots, h)$. A version of Taylor's theorem reads (refer to Abraham and Robbin [1] and Dieudonné [4]):

Taylor's theorem. *If $f(x) \in C^k(E,F)$ where E and F are Banach spaces, then*

$$f(x+h) - f(x) - \sum_{i=1}^{k} \frac{D^if(x)[h]}{i!}$$
$$= \int_0^1 \frac{(1-t)^{k-1}}{(k-1)!}(D^kf(x+th) - D^kf(x))[h]dt\,.$$

Proposition 1. *If $f \in B^k_M(E,F)$, then*

$$\left\| f(x+h) - f(x) - \sum_{j=1}^{k} D^jf(x)[h]/j! \right\| \leq M\|h\|^{k+1}/(k+1)!\,. \tag{1}$$

Proof. Immediate from Taylor's theorem.

Proposition 2. $B^k_M(E,F) = \{f \mid$ *1) f is bounded on some open set, 2) for every finite dimensional linear subspace H, $f|_H(x)$ is continuous, 3) letting $\Delta_hf(x) = f(x+h) - f(x)$, $\|\Delta_h^{k+1}f(x)\| \leq M\|h\|^{k+1}$ for all x and h in E*$\}$.

Proof. Suppose $f \in B^k_M(E,F)$. By the mean value theorem, we have

$$\begin{aligned}\Delta_h^{k+1}f(x) &= \Delta_h\Delta_h^kf(x) = \Delta_h^kDf(x+c_1h)[h] \\ &= \cdots = \Delta_hD^kf(x+c_1h+\cdots+c_kh)[h]\end{aligned} \tag{2}$$

for some $0 < c_i < 1$. So $\|\Delta_h^{k+1}f(x)\| \leq M\|h\|^{k+1}$.

Suppose that $f(x)$ satisfies the conditions on the right side of (2). For any finite dimensional linear subspace H, find a measure μ_H on H and a $\varphi_{H,n} \in C^\infty(H, R)$ with $\int \varphi_{H,n} d_{\mu_H} = 1$, $\varphi_{H,n} \geq 0$ and $\|y\| > 1/n \Rightarrow y \notin$ support $\varphi_{H,n}$. Define $f_{H,n}(x)$ by $f_{H,n}(x) = \int f(x + y)\varphi_{H,n}(y) d_{\mu_H}(y)$. Then

$$f_{H,n}(x + h) - f_{H,n}(x) = \int (x + y)(D\varphi_{H,n}(y)[-h] + o(\|h\|)) d_{\mu_H}(y) ,$$

so

$$\|f_{H,n}(x + h) - f_{H,n}(x) - \int f(x + y) D\varphi_{H,n}(y)[- h] d_{\mu_H}(y)\| = o(\|h\|) ,$$

and $Df_{H,n}(x)[h] = \int f(x + y) D\varphi_{H,n}(y)[-h] d_{\mu_H}(y)$. Repeating this argument gives $f_{H,n} \in C^\infty(H, F)$. Now $\lim_n f_{H,n}(x) = f(x)$ for $x \in H$, and

$$\|\Delta_h^{k+1} f_{H,n}(x)\| = \left\| \int \Delta_h^{k+1} f(x + y)\varphi_{H,n}(y) d_\mu(y) \right\| \leq M\|h\|^{k+1} .$$

So by (2) we have $\sup_x \|D^{k+1} f_{H,n}(x)\| \leq M$ and $f_{H,n} \in B_M^k(H, F)$, and $D^i f_{H,n}(x)$ is uniformly equicontinuous on bounded sets in H for $i \leq k$. By the Ascoli-Arzela theorem, there are a subsequence m of n and a $d_H^i f(x) \in L_s^i(H, F)$ with $\lim_m D^i f_{H,m}(x) = d_H^i f(x)$. Using Proposition 1 and taking $m \to \infty$ we obtain $\|f(x + h) - f(x) - \sum_{i=1}^k d_H^i f(x)[h]/i!\| \leq M\|h\|^{k+1}/(k + 1)!$.

For any other finite dimensional H', $d_{H'}^i f(x)[h] = d_H^i f(x)[h]$ if $x, x', h \in H \cap H'$, so we have maps $d^i f(x)$ i-multilinear from E to F at each x with

$$\left\| f(x + h) - f(x) - \sum_{i=1}^k d^i f(x)[h]/i! \right\| \leq M\|h\|^{k+1}/(k + 1)! .$$

Suppose that f is bounded near x_0. Find δ such that $\|f(y)\| \leq B$ when $\|y - x_0\| \leq \delta$. Then for $\|h\| = 1$ we have

$$\left\| f\left(x_0 + \frac{\delta h i}{k}\right) - f(x_0) - \sum_{j=1}^k \frac{d^j f(x)}{j!}\left[\frac{\delta h i}{k}\right] \right\|$$
$$\leq \frac{1}{(k + 1)!} M\left(\frac{\delta i}{k}\right)^{k+1} \leq \frac{M\delta^{k+1}}{(k + 1)!} ,$$

so $\|\sum_{j=1}^k (i/k)^j d^j f(x)[\delta h]/j!\| \leq 2B + M\delta^{k+1}/(k + 1)!$. Since the $k \times k$ matrix $A_{ij} = (i/k)^i/j!$ is invertible, $\|d^j f(x)[h]\| \leq k\|A^{-1}\|(2B + M\delta^{k+1}/(k + 1)!)/\delta^j$, and so $d^i f(x_0)$ is bounded at x_0 for $i = 1, \cdots, k$. Now $f_{H,m} \in B_M^k(E, F)$, so $\|D^k f_{H,m}(x + h)[h'] - D^k f_{H,m}(x)[h']\| \leq M\|h\| \|h'\|^{k+1}$ for x, h, $h' \in H$. Using the fact that $d^k f(x_0)$ is bounded at x_0 and taking limits over m give

$d^k f(x) \in B^0_M(E, L^k_s(E, F))$. Now $d^i f(x+h) - d^i f(x) = \operatorname{Lim}_m D^i f_{H,m}(x+h) - D^i f_{H,m}(x) = \operatorname{Lim}_m \int_0^1 D^{j+1} f_{H,m}(x+th)[h]dt$. By the uniform convergence of $D^{j+1} f_{H,m}(x+th)$ on $0 \le t \le 1$, this is equal to $\int_0^1 d^{j+1} f(x+th)[h]dt$. Thus $d^j f(x+h) - d^j f(x) = \int d^{j+1} f(x+th)[h]dt$, and by taking $j = k-1, k-2, \cdots, 0$ we have $Dd^j f(x) = d^{j+1} f(x)$ and $f(x) \in B^k_M(E, F)$ with $D^j f = d^j f$.

Proposition 3. *Suppose $f_p \in B^k_M(E, F)$ and $\operatorname{Lim}_p f_p(x) = f(x)$ for all x in E. If f_p are uniformly bounded on some open set, then $f \in B^k_M(E, F)$ and $D^j f(x)[h] = \operatorname{Lim}_p D^j f_p(x)[h]$.*

Proof. The $f_p|_H(x)$ are uniformly equicontinuous on bounded sets in a finite dimensional linear subspace H of E, so $f|_H(x)$ is continuous. Also

$$\|\Delta_h^{k+1} f(x)\| = \|\operatorname{Lim}_p \Delta_h^{k+1} f_p(x)\| \le M \cdot \|h\|^{k+1}.$$

By Proposition 2, $f \in B^k_M(E, F)$. Using (2) we have $D^j f(x)[h] = \operatorname{Lim}_{t\to 0} \Delta^j_{th} f(x)/t^j = \operatorname{Lim}_{t\to 0} \operatorname{Lim}_p \Delta^j_{th} f_p(x)/t^j = \operatorname{Lim}_p \operatorname{Lim}_{t\to 0} \Delta^j_{th} f_p(x)/t^j = \operatorname{Lim}_p D^j f_p(x)[h]$ by the uniform convergence of $\operatorname{Lim}_{t\to 0} \Delta^j_{th} f_p(x)/t^j$ in p.

Proposition 4 (*Inverse Taylor's theorem*). *Suppose $f: E \to F$ is bounded on some open set, and for all x there are maps $d^j f(x)$: j-multilinear from E to F satisfying*

$$(3) \qquad \left\| f(x+h) - f(x) - \sum_{j=1}^{k} d^j f(x)[h]/j! \right\| \le M \cdot \|h\|^{k+1}/(k+1)!\,.$$

Then $f \in B^k_M(E, F)$ and $D^j f(x) = d^j f(x)$.

Proof. For any x and h, $\|f(x+ph) - f(x) - \sum_{j=1}^k p^j d^j f(x)[h]/j!\| \le M \cdot p^{k+1}\|h\|^{k+1}$. Also $\sum_{p=0}^{k+1} (-1)^p \binom{k+1}{p} p^j = 0$ for $0 \le j \le k$, so multiplying the first equations by $(-1)^p \binom{k+1}{p}$ and adding from $p = 0, \cdots, k+1$ give $\|\Delta_h^{k+1} f(x)\| = \left\| \sum_{p=0}^{k+1} (-1)^p \binom{k+1}{p} f(x+ph) \right\| \le M \sum_{p=0}^{k+1} \binom{k+1}{p} p^{k+1} \|h\|^{k+1}$. Hence by Proposition 2, $f \in B^k(E, F)$ and $D^j f(x) = d^j f(x)$. Suppose $x, h, h' \in$ a finite dimensional linear subspace H, and let $f_{H,n} = \int f(x+y)\varphi_{H,n}(y)d_{\mu_H}(y)$ as in Proposition 2. Then $f_{H,n}$ satisfies (3) with $D^j f_{H,n} = \int D^j f(x+y)\varphi_{H,n}(y)d_{\mu_H}(y)$ and so $\|D^{k+1} f_{H,n}\| \le M$. Thus $f_{H,n} \in B^k_M(H, F)$, and $\|D^k f(x+h)[h'] - D^k f(x)[h']\| = \operatorname{Lim}_n \|D^k f_{H,n}(x+h)[h'] - D^k f_{H,n}(x)[h']\| \le M\|h\| \cdot \|h'\|^k$. So $f \in B^k_M(E, F)$. q.e.d.

By proposition 2 we can characterize $B^k_M(E,F)$ without mentioning the derivatives.

Even though at every x, $f(x) = \operatorname{Lim}_p f_p(x)$ in norm, $D^j f_p(x)$ need not approach $D^j f(x)$ in norm as the example $f_n(x) = \langle e_n, x\rangle$ where e_n is an orthonormal basis in l^2 and $f(x) = 0$ shows.

Corollary 1. *For any real number b and open U in E, $X = B^k_M(E,F) \cap \{f \mid \|f(x)\| \le b$ for $x \in U\}$ is compact in the topology of pointwise convergence on E to the weak topology on F.*

Proof. Let $b(x) = \sup_{f \in X} \|f(x)\|$. Then by Proposition 3, $B^k_M(E,F) \cap \{f \mid \|f(x)\| \le b$ for $x \in U\}$ is closed in the compact $\prod_{x \in E} b(x) \subset F^E$.

Corollary 2. $B^1_M(E, F) = \{f \mid f(x) \in C^0(E, F)$ *and* $\|f(x+h) + f(x-h) - 2f(x)\| \le M\|h\|^2\}$.

Remarks. The class $B^k(E,F)$ may be extended to a class $U^k(E,F) = \{f \mid f \in C^k(E,F)$ and for every x in E there are a neighborhood U of x and a M such that $f|_U \in B^k_M(U,F)\}$. Then $C^{k+1}(E,F) \subset U^k(E,F) \subset C^k(E,F)$, and Propositions $1, \cdots, 4$ have obvious generalizations to $U^k(E,F)$.

Theorem 1. *Suppose that E is a B^p smooth separable Banach space, and $\{U_\alpha\}$ is an open cover. Then there exists a partition $\{f_i\}$ of unity refining $\{U_\alpha\}$ with $f_i \in B^p(E,R)$ for each i.*

Proof. We find two countable locally finite open covers $\{V^1_i\}$, $\{V^2_i\}$ refining $\{U_\alpha\}$ and maps $g_i \in B^p(E,R)$ such that $\bar V^1_i \subset V^2_i$, $0 \le g_i(x) \le 1$, $g_i(\bar V^1_i) = 1$ and $g_i(CV^2_i) = 0$. For every $x \in E$ find a $\varphi_x \in B^p(E,R)$ such that $0 \le \varphi_x \le 1$, $\varphi_x(x) = 1$ and that support φ_x is contained in some U_α. Let $A_x = \{y \mid \varphi_x(y) > 1/2\}$. Then $\{A_x\}$ covers E and, since E is Lindelof, we can extract a countable subset $\{A_{x_i}\}$ of $\{A_x\}$ which also covers E. Now let $B_j = \{t_j \ge 1/2; t_i \le 1/2 + 1/j, i < j\}$, $C_j = \{t_j \le 1/2 - 1/j$, or $t_i \ge 1/2 + 2/j$, for some $i < j\}$ in R^j. Then distance $(B_j, C_j) > 0$, and we can find $\eta_j \in B^p(R^j, R)$, with $\eta_j(t_1, \cdots, t_j) = 1$ for $(t_1, \cdots, t_j) \in B_j$ and $\eta(t_1, \cdots, t_j) = 0$ for $(t_1, \cdots, t_j) \in C_j$. Let $\psi_1(x) = \varphi_{x_1}$ and $\psi_j(x) = \eta_j(\varphi_{x_1}(x), \cdots, \varphi_{x_j}(x))$ for $j \ge 2$. Define $V^1_i = \{x \mid \psi_i(x) > 1/2\}$, $V^2_i = \{x \mid \psi_i(x) > 0\}$. Since $V^2_i \subset$ support φ_{x_i}, $\{V^2_i\}$ refines $\{U_\alpha\}$. To show that $\{V^1_i\}$ covers E, suppose that $x \in E$ and that $i(x)$ is the first integer for which $\varphi_i(x) \ge 1/2$. Such an integer exists because the A_i's cover E. Then $\psi_{i(x)} = 1$, and hence $x \in V^1_{i(x)}$, so $\{V^1_i\}$ covers E. Now again suppose that $x \in E$ and find an integer $n(x)$ such that $\varphi_{n(x)}(x) > 1/2$. Then there exist, by the continuity of $\varphi_{n(x)}$, a neighborhood N_x of x and an $a_x > 1/2$ such that $\inf_{y \in N_x} \varphi_{n(x)}(y) \ge a_x$. Pick k large enough so that $k > n(x)$ and $2/k < a_x - 1/2$. Then for $j \ge k$, $\varphi_{n(x)}(y) > 1/2 + 2/j$ for $y \in N_x$, and hence $\psi_j(y) = 0$ for $y \in N_x$. Therefore $N_x \cap V^2_j = \emptyset$ for $j \ge k$ so that $\{V^2_i\}$ is locally finite. Finally take some $h \in B^p(R,R)$ with $h(t) = 1$ for $t \le 0$ and $h(t) = 0$ for $t \ge 1/2$, $0 \le h \le 1$. Defining $g_i(x) = h(\psi_i(x))$ we have that $g_i \in B^p(E,R)$ and $0 \le g \le 1$, $g_i(\bar V^1_i) = 1$ $g_i(CV^2_i) = 0$. Now let $f_1(x) = g_1(x)$ and $f_i(x) = g_i(x)(1 - g_1(x)) \cdots (1 - g_{i-1}(x))$ for $i \ge 2$. Then $f_i \in B^p(E,R)$ and support $f_i \subset$ support $g_i \subset V^2_i$, hence every point of E has a neighborhood on all but a finite number of f_i's vanish. Since $\{x|g_i(x) = 1\} \supset V_i^2$, $\prod_{i=1}^n (1 - g_i(x)) = 0$ for every x and some n. Also $\sum_{i=1}^n f_i(x) = 1 - \prod_{i=1}^n (1 - g_i(x))$, so $\sum_{i=1}^n f_i(x) \equiv 1$ and $\{f_i\}$ is a partition of unity refining $\{U_\alpha\}$ with $f_i \in B^p$ for each i. q.e.d.

For the $\mathscr{L}^p$ spaces it can be shown that for p an even integer $D^{p+1}\|x\|^p = 0$ and that for p not an even integer $\|D^k\|x + h\|^p - D^k\|x\|^p\| \leq (p!/k!)\|h\|^{p-k}$ (see Bonic and Frampton [2]). So $\mathscr{L}^p$ is B^∞ smooth for p an even integer and $\mathscr{L}^p$ is $B^{[p-1]}$ smooth for p not an even integer. Not every C^1 smooth space is B^1 smooth as the following corollary shows (see also Wells [10]).

Theorem 2. *If $n = 2^N$, endow n-dimensional Euclidean space E^n with the norm $\|x\| = \sup_{i=1,\cdots,n} |x_i|$. Suppose $f \in B_M^1(E^n, R)$ with $f(0) = 0$ and $f(x) \geq 1$ when $\|x\| \geq 1$. Then $M \geq 2N$.*

Proof. Assume $M < 2N$, and let $A = \{x|x_i = \pm 1/N$ for $i = 1, \cdots, n$ except for at most one i_0 where $|x_{i_0}| \leq 1/N\}$. Then A is radially symmetric and connected, so there is an $h_1 \in A$ with $Df(0)[h_1] = 0$. h_1 has at least 2^{N-1} components $= 1/N$. Likewise there is an $h_2 \in A$ with $Df(h_1)[h_2] = 0$, and we can choose $\sigma_2 = \pm 1$ so that $h_1 + \sigma_2 h_2$ has at least 2^{N-2} components equal to $2/N$. Inductively choose $h_k \in A$ and σ_k, $k = 3, \cdots, N$, such that $Df(h_1 + \sigma_2 h_2 + \cdots + \sigma_{k-1}h_{k-1})[h_k] = 0$ and that $h_1 + \sigma_2 h_2 + \cdots + \sigma_k h_k$ has 2^{N-k} components equal to k/N. Then $\|h_1 + \cdots + \sigma_N h_N\| = 1$ so by Proposition 1,

$$\begin{aligned} |1 - 0| &= |f(h_1 + \sigma_2 h_2 + \cdots + \sigma_N h_N) - f(0)| \\ &= \sum_{k=1}^N |f(h_1 + \sigma_2 h_2 + \cdots + \sigma_k h_k) - f(h_1 + \sigma_2 h_2 + \cdots + \sigma_{k-1} h_{k-1})| \\ &\leq N \cdot \tfrac{1}{2} M N^{-2} < 1 , \end{aligned}$$

a contradiction.

Corollary 3. *c_0 is not B^1 smooth.*

Proof. Assume $f \in B_M^1(c_0, R)$ with $f(0) = 0$ and $f(1) \geq 1$ when $\|x\| \geq 1$, and restrict f to $\{x|x_i = 0, i > 2^{(M+1)/2}\}$ to get a contradiction to the theorem.

Remark. In this theorem we have only used the uniform continuity of Df.

3. Convex sets and B_M^1 functions

If A is a subset of a Banach space E, let $d(x, A) = \inf_{y \in A} \|y - x\|$. Then $d(x, A) \in B_1^0(E, R)$. If A is convex, $d(x, A)$ shares many of the properties of $\|x\|$. The first proposition is well-known. See Restrepo [8] or Phelps [7].

Proposition 5. *Let A be a closed convex subset of a Banach space with norm differentiable away from zero. Suppose that $d(x, A) = \|x - p(x)\|$ for every x in E and some $p(x)$ in A. Then $d(x, A) \in D(E - A, R)$ and $Dd(x, A) = D\| \|(x - p(x))$.*

Proof. Let $D\| \|(x)$ denote the derivative of $\| \|$ at x. Then for $x \in A$, $\|x + h - p(x)\| = \|x - p(x)\| + D\| \|(x - p(x))[h] + o(\|h\|)$, and for any h

with $p(x) + h \in A$, $\|x - (p(x) + h)\| \geq \|x - p(x)\|$ which implies $D\| \|(x - p(x))[h] \leq 0$. Thus the hyperplane $L = \{y | D\| \|(x - p(x))[y - p(x)] = 0\}$ is a supporting hyperplane for A at $p(x)$, and $d(x + h, L) \leq d(x + h, A) \leq d(x + h, p(x))$ so that

$$\|x - p(x)\| + D\| \|(x - p(x))[h] \\ \leq d(x + h, A) \leq \|x - p(x)\| + D\| \|(x - p(x))[h] + o(\|h\|) .$$

Hence $0 \leq d(x + h, A) - d(x, A) - D\| \|(x - p(x))[h] \leq o(\|h\|)$, and so $d(x, A)$ is differentiable at x and $Dd(x, A) = D\| \|(x - p(x))$.

Proposition 6. *If A is closed and convex and $\|x\| \in B^1_{M/\alpha}(\{x | \|x\| > \alpha\}, R)$, then $d(x, A) \in B^1_{M/\alpha}(\{x | d(x, A) > \alpha\}, R)$.*

Proof. Suppose that every point x in E has a closest point $p(x)$ in A. By Proposition 1, if $d(x, p(x)), d(x + h, p(x)) > \alpha$, then $|d(x + h, p(x)) - d(x, p(x)) - D\| \| (x - p(x))[h]| \leq \frac{1}{2}M\|h\|^2/\alpha$, and we have

$$0 \leq d(x + h, A) - d(x, A) - D\| \|(x - p(x))[h] \leq \tfrac{1}{2}M\|h\|^2/\alpha$$

by arguing as in Proposition 5, and therefore $d(x, A) \in B^1_{M/\alpha}(\{x | d(x, A) > \alpha\}, R)$ by Proposition 4. Now suppose that A is arbitrary. If H is a finite dimensional linear subspace, then every point in E has a closest point in $A \cap H$. Hence $d(x, A \cap H) \in B^1_{M/\alpha}(\{x | d(x, A) > \alpha\}, R)$. With the finite dimensional linear subspaces ordered by inclusion, $d(x, A) = \operatorname{Lim}_H d(x, A \cap H) \in B^1_{M/\alpha}(\{x | d(x, A) >(\alpha\}, R)$ by Proposition 3.

Proposition 7. *Suppose that A is a closed convex subset of E and that $\|x\|^2 \in B^1_M(E, R)$. Then $d^2(x, A) \in B^1_M(E; R)$.*

Proof. Suppose every point x of E has a closest point $p(x)$ of A. Then

$$d^2(x + h, A) \leq \|x + h - p(x)\|^2 \\ \leq \|x - p(x)\|^2 + D\| \|^2(x - p(x))[h] + \tfrac{1}{2}M\|h\|^2.$$

Defining $L = \{y | D\| \|(x - p(x))[y - p(x)] = 0\}$ gives

$$d^2(x + h, A) \geq d^2(x + h, L) = (\|x - p(x)\| + D\| \|(x - p(x))[h])^2 \\ \geq \|x - p(x)\|^2 + 2D\| \|(x - p(x))[h](\|x - p(x)\|) \\ = \|x - p(x)\|^2 + D\| \|^2(x - p(x))[h],$$

so $|d^2(x + h, A) - d^2(x, A) - D\| \|^2(x - p(x))[h]| \leq \frac{1}{2}M\|h\|^2$. Thus $d^2(x, A) \in B^1_M(E, R)$ by Proposition 4. Taking limits of $d^2(x, A \cap H)$ over finite dimensional linear spaces H gives as above $d^2(x, A) \in B^1_M(E, R)$ for arbitrary A.

Remarks. If E happens to be uniformly convex, then every point x has a closest point $p(x)$ in a closed convex A and $p(x)$ is continuous. So, if $\|x\| \in C^1(E - \{0\}, R)$, then $d(x, A) \in C^1(E - A, R)$. The question of whether $\|x\| \in C^1(E - \{0\}, R)$ implies $d(x, A) \in C^1(E - A, R)$ in general remains open.

4. B^1 functions on Hilbert space

We will suppose that H is a real Hilbert space endowed with the usual norm, and we will identify H^* with H and write $\langle y, x\rangle = y\cdot x$ and $\|x\|^2 = x^2$.

We recall the Whitney extension theorem (see Abraham and Robbin [1]): Let $A \subset R^n$ be a closed subset, and $f_i, i = 0, \cdots, k$: $A \to L^i_s(R^n, F)$, F another Banach space, and suppose

$$\operatorname*{Lim}_{x,y\to x_0;\ x,y,x_0\in A} \|f_j(y) - \textstyle\sum_{i=j}^{k} f_i(x)[y-x]/(i-j)!\|/\|x-y\|^{k-j} = 0 .$$

Then f_0 has a C^k extension to R^n with $D^j f_0(x) = f_j(x)$ for $x \in A$.

In this section we prove a version of this for real-valued B^1 functions on Hilbert space, and show that C^1 partitions of unity exist on any non-separable Hilbert space.

Theorem 1. *Let $A = \{p_1, \cdots, p_m\}$ be a finite subset of R^n endowed with the usual norm. Let $a_{p_i} \in R, y_{p_i} \in R^n$ for $i = 1, \cdots, m$ satisfy*

$$(4)\qquad a_{p'} \le a_p + \tfrac{1}{2}(y_p + y_{p'})\cdot(p' - p) + \tfrac{1}{4}M(p' - p)^2 - \tfrac{1}{4}(y_{p'} - y_p)^2/M$$

for all p, p' in A. Then there exists an $f(x) \in B^1_M(R^n, R)$ with $f(p) = a_p, Df(p) = y_p$ for p in A and $f(x) \ge \inf_{p\in A} [a_p - \frac{1}{2}y_p^2/M + \frac{1}{4}M(x - p + y_p/M)^2]$. Further, if $g(x) \in B^1_M(R^n, R)$ with $g(p) = a_p, Dg(p) = y_p$ when $p \in A$, then $g(x) \le f(x)$ for all x.

Proof. We first construct a convex linear cell complex and a dual complex. From these a cellular decomposition of R^n is constructed on which f is defined. Df will turn out to be piecewise linear.

Definition. When $p \in A$ we define:

$$\tilde{p} = p - y_p/M , \qquad \overline{p} = \{p' | \tilde{p}' = \tilde{p}, p' \in A\} ,$$
$$d_p(x) = a_p - \tfrac{1}{2}y_p^2/M + \tfrac{1}{4}M(x - \tilde{p})^2 .$$

Definition. When $S \subset A$ we define:

$$d_S(x) = \inf_{p\in S} d_p(x) , \quad \tilde{S} = \{\tilde{p} | p \in S\} ,$$
$$S_H = \text{smallest hyperplane containing } \tilde{S} ,$$
$$S_E = \{x | d_p(x) = d_{p'}(x) \text{ for all } p, p' \in S\} ,$$
$$S_* = \{x | d_p(x) = d_{p'}(x) \le d_{p''}(x) \text{ for all } p, p' \in S, p'' \in A\} ,$$
$$K = \{S | S \subset A \text{ and for some } x \in S_*, d_S(x) < d_{A-S}(x)\} .$$

So, if $p \in S \in K$ then $\overline{p} \subset S$.

Definition. $\hat{S}$ = convex hull of $\tilde{S}$.

Lemma 1. $\{p, p'\}_E = R^n$ *or an* $(n-1)$*-dimensional hyperplane, and* $\tilde{p}' - \tilde{p} \perp \{p, p'\}_E$.

Proof. $d_p(x) - d_{p'}(x) = (a_p - \frac{1}{2}y_p^2/M) - (a_{p'} - \frac{1}{2}y_{p'}^2/M) + \tilde{p}^2 - \tilde{p}'^2 +$

$2x\cdot(\tilde{p}' - \tilde{p})$. If $\tilde{p} \neq \tilde{p}'$, this immediately gives the lemma. If $\tilde{p} = \tilde{p}'$, then $p - p' = (y_p - y_{p'})/M$ and (4) gives $a_{p'} - \frac{1}{2}y_{p'}^2/M \leq a_p - \frac{1}{2}y_p^2/M$. Reversing p and p' gives $d_p(x) = d_{p'}(x)$.

Lemma 2. *S_* is closed and convex.*

Proof. By the definition and Lemma 1, S_* is the intersection of closed convex sets.

Definition. Let $\hat{S}^b, S_*^b$ be the relative boundaries of $\hat{S}, S_*$ if $\dim \hat{S}$, $\dim S_* \neq 0$, in which case let $\hat{S}^i, S_*^i$ be the relative interiors; if $\dim \hat{S}$, $\dim S_* = 0$, let $\hat{S}^b = \emptyset$, $S_*^b = \emptyset$, $\hat{S}^i = \hat{S}$ and $S_*^i = S_*$.

Lemma 3. *$S_H \perp S_E$, and if $S_E \neq \emptyset$, then* $\dim S_H + \dim S_E = n$.

Proof. $S_E = \bigcap_{p,p' \in S} \{p, p'\}_E$ together with Lemma 1 implies $S_H \perp S_E$. Assume $\dim S'_H + \dim S'_E = n$ for $S' = S - \overline{p}$, and $p' \in S - \overline{p}$. Then by Lemma 1 $\dim (p, p')_H + \dim (p, p')_E = n$, and $\dim S_E = \dim (\{p, p'\}_E \cap (S - \overline{p})_E) = n - \dim (\{p, p'\}_H \cup (S - \overline{p})_H) = n - \dim S_H$. By induction $\dim S_H + \dim S_E = n$ for all S.

Lemma 4. *If $S \subset S'$, then $S'_* \subset S_*$. If $S, S' \in K$, then $S \subsetneq S'$ if and only if $S'_* \subsetneq S_*$, and $S = S'$ if and only if $S_* = S'_*$.*

Proof. The first statement follows from the definition of S_*. If $S \subsetneq S'$ and $S, S' \in K$, find $z \in S_*$ with $d_S(z) < d_{S'-S}(z)$ so that $S_* \neq S'_*$ and $S'_* \subsetneq S_*$. If $S'_* \subset S_*$, find $z \in S'_*$ with $d_{S'}(z) < d_{A-S'}(z)$. So, if $p \in S$, then $d_{S'}(z) = d_p(z)$, so $p \notin A - S'$, and hence $S \subset S'$.

Lemma 5. *If $S \in K$, then $S_*^i = \{x | x \in S_E, d_S(x) < d_{A-S}(x)\}$.*

Proof. If $S \in K$, then clearly $\{x | x \in S_E, d_S(x) < d_{A-S}(x)\} \subset S_*^i$. Suppose $x \in S_*^i$ and $p \in S$, $p' \in A$ with $d_p(x) = d_{p'}(x)$. Then the hyperplane $d_p(x) = d_{p'}(x)$ must contain all of S_*, so $p' \in S$. Therefore $d_S(x) < d_{A-S}(x)$.

Lemma 6. $d_{p'}(y') - d_p(y') = d_{p'}(y) - d_p(y) + 2(y' - y)\cdot(\tilde{p} - \tilde{p}')$.

Proof. Immediate from the definition.

Lemma 7. *$\hat{S} \perp S_*$. For $S \in K$,* $\dim \hat{S} + \dim S_* = n$.

Proof. $S_H \perp S_E$ implies the first part. Suppose $S \in K$ and find $z \in S_*$ with $d_S(z) < d_{A-S}(z)$. But then for some ε, open ball center z radius $\varepsilon \cap S_E \subset S_*$ so $\dim S_* = \dim S_E$ and $\dim \hat{S} + \dim S_* = n$.

Definition. If $S_* \neq \emptyset$, let $\overline{\overline{S}} = \{p | p \in A, d_p(z) = d_S(z) \text{ for all } z \in S_*\}$.

Lemma 8. *If $S_* \neq \emptyset$, then $\overline{\overline{S}} \in K$ and $\overline{\overline{S}}_* = S_*$.*

Proof. Immediate from the definitions.

Lemma 9. (a) *If $S, S' \in K$ and $S \cap S' \neq \emptyset$, then $S \cap S' \in K$ and $\hat{S} \cap \hat{S}' = \widehat{S \cap S'}$.*

(b) *If $S, S' \in K$ and $S_* \cap S'_* \neq \emptyset$, then $S_* \cap S'_* = (\overline{\overline{S \cup S'}})_*$.*

Proof. (a) Assume $S \not\subset S'$ and $S' \not\subset S$, and find $y \in S_*$, $y' \in S'_*$ with $d_S(y) < d_{A-S}(y)$, $d_{S'}(y') < d_{A-S'}(y')$. Then $L = \text{cohull } \{y, y'\} \subset (S \cap S')_E$. For any $p' \in A - (S \cup S')$ and $p \in S \cap S'$, the half space $d_p(x) \geq d_{p'}(x)$ does not contain y or y', so it does not contain L. For $p \in S \cap S'$ and $p' \in (S - S') \cup$

$(S' - S)$, the half space $d_p(x) \geq d_{p'}(x)$ does not contain both y and y'. Since $d_{p'}(y) = d_p(y)$ or $d_{p'}(y') = d_p(y')$, the half space $d_p(x) \geq d_{p'}(x)$ can not intersect L^i. Picking $z \in L^i$ we have $d_{S \cap S'}(z) < d_{A - S \cap S}(z)$, so $S \cap S' \in K$. $\hat{S} \cap \hat{S}' = \widehat{S \cap S'}$ is obvious.

(b) Observe $(S \cup S')_* = S_* \cap S'_*$ and use Lemma 8.

Lemma 10. *If* $S \in K$, *then* $\hat{S}^b = \bigcup_{S' \subsetneq S, S' \in K} \hat{S}'$.

Proof. Suppose $x \in \hat{S}^b$. Then $x \in \hat{S}'$ for some $S' \subset S$ with $\hat{S}' \subset \hat{S}^b$. Find an $(n-1)$-dimensional hyperplane M containing $\hat{S}'$, supporting the convex set $\hat{S}$ but not containing $\hat{S}$. Find $y \in S_*$ with $d_S(y) < d_{A-S}(y)$, and find $y' \neq y$ with $y - y' \perp M$, with $\hat{S}$ on the side of M in direction y' to y and $d_{S'}(y') < d_{A-S}(y')$. Then $y' - y \perp (S \cap M)_H$ which implies $y' \in S'_E$. For all $p' \in S'$ and $p \in S$, $(\tilde{p}' - \tilde{p}) \cdot (y - y') \geq 0$. Thus $d_p(y') \geq d_{p'}(y')$ by Lemma 6 so that $d_{S'}(y') \leq d_S(y')$. Also $(\tilde{p} - \tilde{p}') \cdot (y - y') > 0$ for some $p \in S - M$ and all $p' \in S'$, so by Lemma 6, $d_p(y') > d_{p'}(y')$; hence $d_{S'}(y') < d_p(y')$. Thus $y' \in S'_*$ and $y' \notin S_*$. So $\bar{\bar{S}}'_* = S'_* \supsetneq S_*$, and $x \in \hat{S}' \subset \hat{\bar{\bar{S}}}' \subsetneq \hat{S}$ with $\bar{\bar{S}}' \in K$ by Lemmas 8 and 4.

Suppose on the other hand that $S' \subsetneq S$, $S \in K$. Then $S'_* \supsetneq S_*$ by Lemma 4, so we can find $y' \in S'^i_* - S_*$ and $y \in S_*$. Take some $p' \in S'$, and let $M = \{x | (y - y') \cdot (x - \tilde{p}') = 0\}$. Then $y, y' \in S'_E$, and $S'_H \subset M$ by Lemma 2 and so $\hat{S}' \subset M$. Now if $p \in S - S'$, then $d_{p'}(y') < d_p(y')$. Since $d_{p'}(y) = d_p(y)$, $(y - y') \cdot (\tilde{p} - \tilde{p}') > 0$ by Lemma 6 and $\tilde{p}$ lies on the side of M in direction y' to y. Hence M supports $\hat{S}$ and $\hat{S}' \subset \hat{S}^b$.

Lemma 11. *If* $S \in K$, *then* $S^b_* = \bigcup_{S' \supsetneq S, S' \in K} S'_*$.

Proof. By Lemma 5, $x \in S^b_*$ if and only if $x \in S'_*$ for some $S' \supsetneq S$. But then $x \in \bar{\bar{S}}'_*$ with $\bar{\bar{S}}' \in K$ and $\bar{\bar{S}}' \supset S' \supsetneq S$.

By Lemmas 9, 10, 11, $\bigcup_{S \in K} \hat{S}$ is a cell complex, and $\bigcup_{S \in K} S_*$ is a cell complex of $R^n \cup \infty$ dual to $\bigcup_{S \in K} \hat{S}$ by Lemma 4. We show that $\bigcup_{S \in K} \hat{S} = \text{cohull } \tilde{A}$. We can assume that $\dim \tilde{A} = n$. Suppose $S \in K$ with $\dim \hat{S} = n - 1$. From Lemma 6 we see that S_* extends infinitely in a half space determined by S_H if and only if there are no points of $\tilde{A}$ in that open half space. Hence $\hat{S} \subset (\text{cohull } \tilde{A})^o$ if and only if S_* has only one boundary point if and only if $\hat{S}$ does not lie on the boundary of two other $\hat{S}$'s in K by Lemma 11. Now there are members of K with $\dim \hat{S} = n$, otherwise if $\dim \hat{S}' = \max_{S \in K} \dim \hat{S} < n$ then $S'_* = S'_E$. However, $p' \in S'$ implies that $\{p', p''\}_E$ must intersect S'_E for some $p'' \in A$, otherwise $\dim A$ would be less than n. Thus $S'_E \neq S_*$, a contradiction. Hence $\emptyset \neq \left(\bigcup_{S \in K, \dim S = n} \hat{S}\right)^o \subset (\text{cohull } \tilde{A})^o$ so that $\bigcup_{S \in K} \hat{S} = \text{cohull } \tilde{A}$.

We also observe that for any x in R^n if we let $S = \{p | d_p(x) = \inf_{p' \in A} d_{p'}(x)\}$ then $S \in K$ and $x \in S_*$. Hence $\bigcup_{S \in K} S_* = R^n$. Fig. 1 shows an example of these two cell complexes.

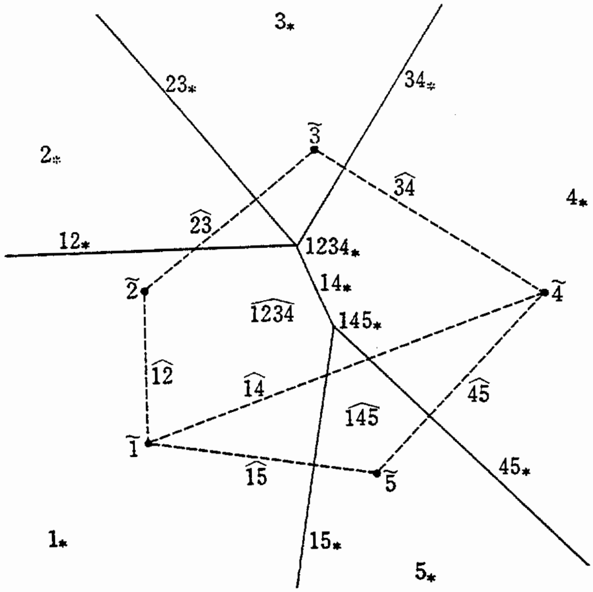

Fig. 1

We now perform another decomposition of R^n.

Definition. For all $S \in K$ let $T_S = \{x \mid x = \frac{1}{2}(y + z)$ for some $y \in \hat{S}$ and $z \in S_*\}$.

Lemma 12. *T_S is closed and convex with nonempty interior.*

Proof. Immediate from Lemmas 2 and 3.

Lemma 13. *The representation $x = \frac{1}{2}(y + z)$, $y \in \hat{S}$, $z \in S_*$ for $x \in T_S$ is unique.*

Proof. Suppose $x = \frac{1}{2}(y' + z')$, $y' \in \hat{S}$, $z' \in S_*$ also. Then $y - y' = -(z - z')$, and $y - y' \perp z - z'$ by Lemma 3, so $y = y'$ and $z = z'$.

Lemma 14. $T_S \cap T_{S'} = \{x \mid x = \frac{1}{2}(y + z), y \in \hat{S} \cap \hat{S}', z \in S_* \cap S'_*\}$.

Proof. Immediate from Lemma 13.

Lemma 15. (a) $(T_S \cap T_{S'})^0 = \emptyset$ *if* $S \neq S'$, *and* (b) $T_S^{\partial} \subset \bigcup_{S, S' \in K, S' \supsetneqq S \text{ or } S' \subsetneqq S} T_{S'}$.

Proof. (a) $\hat{S} \cap \hat{S}' = \widehat{S \cap S'}$, and $S \cap S' \in K$ by Lemma 9. If $S \neq S'$, then $\hat{S} \cap \hat{S}' \subset \hat{S}^b$ or $\hat{S}'^b$, so $\dim(\hat{S} \cap \hat{S}') < \max(\dim \hat{S}, \dim \hat{S}') = n - \min(\dim S_*, \dim S'_*) \leq n - \dim(S_* \cap S'_*)$. By Lemma 14, $\dim(T_S \cap T_{S'}) < n$ and the interior of $T_S \cap T_{S'}$ is empty.

(b) If $x \in T_S^{\partial}$, then $x = \frac{1}{2}(y + z)$ where $y \in \hat{S}^b$ and/or $z \in S_*^b$. So $y \in \hat{S}'$ and/or $z \in S''_*$ for some $S' \subsetneqq S$ and $S'' \supsetneqq S$ by Lemmas 10 and 11. Hence $x \in T_{S'}$ and/or $T_{S''}$ with $S' \subsetneqq S$ and/or $S'' \supsetneqq S$.

Lemma 16. $T_S \cap T_{S'} = T_{S \cap S'} \cap T_{\overline{\overline{S \cup S'}}}$.

Proof. $\hat{S} \cap \hat{S}' = \widehat{S \cap S'} \cap \widehat{\overline{\overline{S \cup S'}}}$, and $S_* \cap S'_* = (S \cap S')_* \cap (\overline{\overline{S \cup S'}})_*$ by Lemma 9, and Lemma 16 follows from Lemma 14.

Lemma 17. $\bigcup_{S \in K} T_S = R^n$.

Proof. Since the complement of a closed convex set is locally connected and $T_S^0 \cap T_{S'}^0 = \emptyset$ if $S \neq S'$, this lemma follows from the next proposition.

Proposition 8. *Let $\{T_i\}$ be locally finite collection of nonempty closed subsets of a connected space E. Suppose that the T_i's have disjoint interiors, $E - T_i$ is connected in some neighborhood of each point of E for each i, and $T_i^0 \subset \bigcup_{j \neq i} T_j$. Then $\bigcup_i T_i = E$.*

Proof. $\bigcup T_i$ is closed since $\{T_i\}$ is locally finite. Suppose that whenever a point y of E is contained in k or less T_i's then $y \in \left(\bigcup_i T_i\right)^0$. If $x \in T_{i_1}, \cdots, T_{i_k}$ and no others, then we can find a neighborhood U about x which meets only $T_{i_1}, \cdots, T_{i_k}$ and such that $U - T_{i_1}$ is connected. Thus $T_{i_2} \cup \cdots \cup T_{i_k}$ is open and closed in $U - T_{i_1}$ by assumption, and so contains all of $U - T_{i_1}$. This implies that $U \subset T_{i_1} \cup \cdots \cup T_{i_k}$ so that $x \in \bigcup_i T_i^0$. The statement is true for $k = 1$, so by induction $x \in \bigcup_i T_i^0$ for all $x \in E$. Hence $\bigcup_i T_i$ is open, and $\bigcup_i T_i = E$ since is connected. q.e.d.

Fig. 2 illustrates the T's superimposed on the dual complexes of Fig. 1.

Definition. $S_C = S_E \cap S_H$ for $S \in K$. S_C is a point by Lemma 3.

We now construct f on R^n.

Definition. $f_S(x) = d_S(S_C) + \frac{1}{2}Md^2(x, S_H) - \frac{1}{2}Md^2(x, S_E)$ for $S \in K$ and $x \in T_S$.

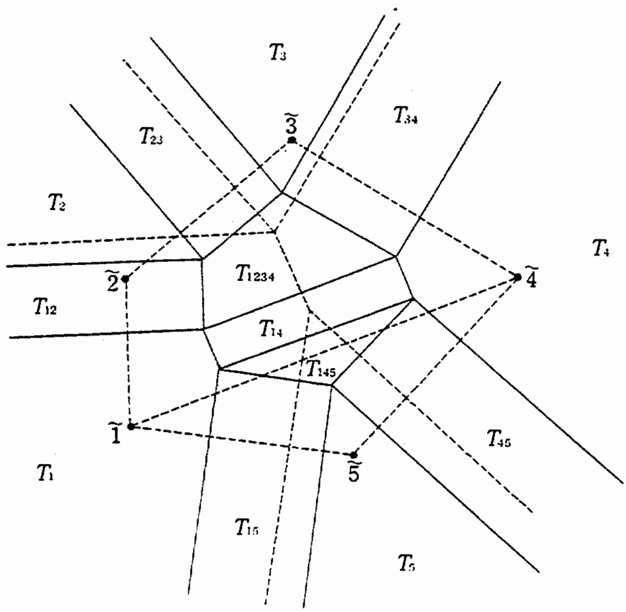

Fig. 2

Lemma 18. $f_S(x) = f_{S'}(x)$ *if* $x \in T_S \cap T_{S'}$.

Proof. By Lemma 16 we can assume that $S \subset S'$. Now $x = \frac{1}{2}(y + z)$ with $y \in \hat{S} \subset \hat{S}'$, $z \in S'_* \subset S_*$ by Lemma 14, and $d(x, S_H) = \frac{1}{2}d(z, S_H) = \frac{1}{2}d(z, S_C)$ and $d(x, S_E) = \frac{1}{2}d(y, S_E) = \frac{1}{2}d(y, S_C)$ so that

$$f_S(x) = d_S(S_C) + \tfrac{1}{8}Md^2(S_C, z) - \tfrac{1}{8}Md^2(S_C, y) ,$$

Similarly,

$$f_{S'}(x) = d_{S'}(S'_C) + \tfrac{1}{8}Md^2(S'_C, z) - \tfrac{1}{8}Md^2(S'_C, y) .$$

Now $z, S'_C \in S'_E$ and $S_C, S'_C \in S'_H$, and $d^2(z, S'_C) + d^2(S_C, S'_C) = d^2(z, S_C)$ since $S'_E \perp S'_H$. Also $y, S_C \in S_H$ and $S_C, S'_C \in S_E$ with $S_H \perp S_E$, so $d^2(y, S_C) + d^2(S_C, S'_C) = d^2(y, S'_C)$. Finally, $\tilde{p} - S_C \perp S_C - S'_C$ for $p \in S$, so $\frac{1}{4}Md^2(S_C, S'_C) + d_S(S_C) = d_S(S'_C) = d_{S'}(S'_C)$. Hence our lemma follows from these equations.

Lemma 19. $f_S \in C^\infty(T_S, R)$.

Proof. Observe that $d^2(x, S_E)$ and $d^2(x, S_H)$ are C^∞.

Lemma 20. $Df(x) = \frac{1}{2}M(z - y)$ *where* $x = \frac{1}{2}(y + z)$, $y \in \hat{S}$ *and* $z \in S_*$.

Proof. $Dd^2(x, S_H) = 2\|x - P_{S_H}(x)\| D\|x - P_{S_H}(x)\| = 2(x - P_{S_H}(x))$ where $P_{S_H}(x)$ is the closest point of S_H to x by Proposition 5. Now $x - P_{S_H}(x) = \frac{1}{2}(z - S_C)$, so $D\frac{1}{2}Md^2(x, S_H) = \frac{1}{2}M(z - S_C)$. Likewise $Dd^2(x, S_E) = \frac{1}{2}M(x - P_{S_E}(x)) = \frac{1}{2}M(y - S_C)$ where $P_{S_E}(x)$ is the closest point of S_E to x. Hence $Df(x) = \frac{1}{2}M(z - y)$.

Lemma 21. $f_S(x) \in B^1_M(T_S, R)$.

Proof. Let $x, x' \in T$, $x = \frac{1}{2}(y + z)$ and $x' = \frac{1}{2}(y' + z')$ as usual. Then by Lemma 20,

$$(Df_S(x) - Df_S(x))^2 = \tfrac{1}{4}M^2((z - z') + (y - y'))^2 = M^2(x - x')^2 ,$$

since $z - z' \perp y - y'$. Hence $\|Df_S(x) - Df_S(x')\| = M\|x - x'\|$.

Lemma 22. $Df_S(x) = Df_{S'}(x)$ *if* $x \in T_S \cap T_{S'}$.

Proof. If $x \in T_S \cap T_{S'}$, then $x = \frac{1}{2}(y + z)$ where $y \in \hat{S} \cap \hat{S}'$ and $z \in S_* \cap S'_*$ by Lemma 14. Thus $Df_S(x) = \frac{1}{2}M(z - y) = Df_{S'}(x)$ by Lemma 21.

Definition. $f(x) = f_S(x)$ if $x \in T_S$.

f is well defined on R^n by Lemmas 17 and 18, and $f \in B^1_M(R^n, R)$ by Lemmas 21 and 22.

Lemma 23. $f(p) = a_p$ *and* $Df(p) = y_p$ *if* $p \in A$.

Proof. By the definition and an assumption in the hypothesis of Theorem 1, for any $p' \in A$ we can easily obtain $d_p(p + y_p/M) = \frac{1}{2}y_p^2/M + a_p \leq \frac{1}{2}y_p^2/M + a'_p + \frac{1}{4}M(p - p')^2 + \frac{1}{2}(y_{p'} + y_p)\cdot(p - p') - \frac{1}{4}(y_p - y_{p'})^2/M = d_{p'}(p + y_p/M)$. Thus $p + y_p/M \in \overline{p}_*$ and $p = \frac{1}{2}(\tilde{p} + p + y_p/M) \in T_{\overline{p}}$. Hence $f(p) = f_{\overline{p}}(p) = d_{\overline{p}}(p) + \frac{1}{2}M(p - (p - y_p/M))^2 = a_p$, $Df(p) = \frac{1}{2}M((p + y_p/M) - \tilde{p}) = y_p$ by Lemma 20.

Lemma 24. *Suppose* $g \in B^1_M(R^n, R)$ *and* $g(p) = a_p$, $Dg(p) = y_p$ *for* $p \in A$. *Then* $g(x) \leq f(x)$.

Proof. Suppose first that $x \in T_{\bar{p}}$. Then by Proposition 1, $g(x) \leq a_p + y_p(x-p) + \frac{1}{2}M(x-p)^2 = a_p - \frac{1}{2}y_p^2/M + \frac{1}{2}M(x-\tilde{p})^2 = f_{\bar{p}}(x) = f(x)$. Suppose next that for all $S \in K$ with $\aleph(\tilde{S}) \leq m$, $g(x) \leq f(x)$ for $x \in T_S$. If $\aleph(\tilde{S}) = m+1$ and $x \in T_S$, then let $x = \frac{1}{2}(y+z)$, $y \in \hat{S}$, $z \in S_*$. Fix z, and define $e(w) = g(\frac{1}{2}(w+z)) - f(\frac{1}{2}(w+z))$ for $w \in \hat{S}$. Then $g(\frac{1}{2}(w+z)) \in B^1_{M/4}(\hat{S}, R)$ and $f(\frac{1}{2}(w+z)) = \text{const.} - \frac{1}{8}M(w-S_C)^2$ with $D_w f(\frac{1}{2}(w+z)) = -\frac{1}{4}M(w-S_C)$. For any h with $w+h \in \hat{S}$, $De(w+h)[h] - De(w)[h] = \frac{1}{4}Mh^2 + (Dg(\frac{1}{2}(w+h+z)) - Dg(\frac{1}{2}(w+z)))[h] \geq 0$. Thus, if $e(w)$ is maximal at w, then $De(w) \neq 0$, so $e(w)$ has its maximum on $\hat{S}^b$. Since $w \in \hat{S}^b$ implies $x \in \hat{S}'$ for some $S' \subsetneq S$, $x = \frac{1}{2}(w+z) \in T_{S'}$ so that $e(w) \leq 0$ by the assumption. Hence $e(w) \leq 0$ on $\hat{S}$ and $g(x) \leq f(x)$ on T_S. By induction $g(x) \leq f(x)$ everywhere.

Lemma 25. $f(x) \geq \inf_{p \in A} d_p(x)$.

Proof. Take p with $d_p(x) = \inf_{q \in A} d_q(x)$. Then $x \in \bar{p}_*$ so $\frac{1}{2}(\tilde{p}+x) \in T_{\bar{p}}$ and $f_{\bar{p}}(\frac{1}{2}(\tilde{p}+x)) = a_p - \frac{1}{2}y_p^2/M + \frac{1}{8}M(x-\tilde{p})^2$. Also $Df(\frac{1}{2}(\tilde{p}+x)) = \frac{1}{2}M(x-\tilde{p})$. So by Proposition 1, $f(x) \geq f(\frac{1}{2}(x+\tilde{p})) + Df(\frac{1}{2}(x+\tilde{p}))[\frac{1}{2}(x-\tilde{p})] - \frac{1}{2}M(\frac{1}{2}(x-\tilde{p}))^2 = d_p(x)$.

Lemmas 24 and 25 complete the proof of Theorem 1. We observe from Lemma 20 that Df is a piecewise linear map from $\bigcup_S T_S$ to R^n, whose derivative in T_S^0 is $M \cdot \text{Identity} \oplus -M \cdot \text{Identity}$ on $S_H \oplus S_E$.

Lemma 26. *Suppose p and $p - y_p/M \in L$ for all p in A where L is an affine linear subspace of R^n. Then $f(x) = f_L(\pi_L(x)) + \frac{1}{2}Md^2(x, L)$, where f_L is the function obtained in Theorem 1 by taking L instead of R^n as the underlying linear space, and π_L is the orthogonal projection of R^n onto L.*

Proof. Observe that $\tilde{p} \in L$ for all p in A and that K is the same taking R^n or L. Also T_S on $R^n = \pi_L^{-1}(T_S \text{ on } L)$, and $d^2(x, S_H) = d^2(\pi_L(x), S_H) + (x - \pi_L(x))^2$, $d^2(x, S_E) = d^2(\pi_L(x), S_E)$. This establishes the lemma.

Theorem 2. *Let A be a closed nonempty subset of any Hilbert space H endowed with the usual norm. Suppose that f_0 is a real-valued function on A. Then there exists an $f \in B^1_M(H, R)$ with $f|_A = f_0$ if and only there is a map $f_1 : A \to H$ such that for all $x, y \in A$*

$$f_0(y) \leq f_0(x) + \tfrac{1}{2}(f_1(x) + f_1(y)) \cdot (y - x) + \tfrac{1}{4}M(y-x)^2 - \tfrac{1}{4}(f(y) - f(x))^2/M \,. \tag{5}$$

Further, f can be found such that $f(x) \geq \inf_{y \in A} d_y(x)$ where $d_y(x) = f_0(y) - \frac{1}{2}f_1^2(y)/M + \frac{1}{4}M(x - y + f_1(y)/M)^2$ and such that if $g(x) \in B^1_M(H, R)$ with $g(x) = f_0(x)$ and $Dg(x) = f_1(x)$ for $x \in A$, then $g(x) \leq f(x)$ for all x.

Proof. If f_0 has an extension f in $B_M(H, R)$, let $f_1(x) = Df(x)$. Let x_i, $i = 0, 1$ be two points in H, set $a_i = f_0(x_i)$ and $y_i = f_1(x_i)$, and define $x_2 = \frac{1}{2}(x_0 + x_1) + \frac{1}{2}(y_1 - y_0)/M$. By Proposition 1 we have

$$f(x_2) \leq f(x_0) + y_0 \cdot (\tfrac{1}{2}(x_1 - x_0) + \tfrac{1}{2}(y_1 - y_0)) + \tfrac{1}{2}M(\tfrac{1}{2}(x_1 - x_0) + \tfrac{1}{2}(y_1 - y_0))^2 \,,$$

$$f(x_2) \geq f(x_1) - y_1 \cdot (\tfrac{1}{2}(x_1 - x_0) - \tfrac{1}{2}(y_1 - y_0)) - \tfrac{1}{2}M(\tfrac{1}{2}(x_1 - x_0) - \tfrac{1}{2}(y_1 - y_0))^2 \,,$$

so by the parallelogram law,

$$\begin{aligned} f(x_1) &\le f(x_0) + \tfrac{1}{2}(y_0 + y_1)\cdot(x_1 - x_0) - \tfrac{1}{2}(y_1 - y_0)^2 \\ &\quad + \tfrac{1}{2}M[2(\tfrac{1}{2}(x_1 - x_0))^2 + 2(\tfrac{1}{2}(y_1 - y_0)/M)^2] \\ &= f(x_0) + \tfrac{1}{2}(y_0 + y_1)\cdot(x_1 - x_0) + \tfrac{1}{4}M(x_1 - x_0)^2 - \tfrac{1}{4}(y_1 - y_0)^2/M . \end{aligned}$$

To go the other way, choose for every finite subset F in A, a finite dimensional linear subspace H_F of H containing p and $p - f_1(p)/M$ for all p in F. By Theorem 1 construct $f'_F \in B^1_M(H_F, R)$ satisfying $f'_F(p) = f_0(p)$, $Df'_F(p) = f_1(p)$ for p in F, etc. Now define for $x \in H$, $f_F(x) = f'_F(\pi_{H_F}(x)) + \frac{1}{2}Md^2(x, H_F)$. Then $f_F \in B^1_M(H, R)$, $f_F(p) = f_0(p)$, $Df_F(p) = f_1(p)$ for $p \in A$, and f_F is independent of H_F by Lemma 26. So we have $f_F(x) \ge \inf_{y \in F} d_y(x)$, and $g(x) \le f(x)$ for all x in H if $g \in B^1_M(H, R)$ with $g(p) = f_0(p)$ and $Dg(p) = f_1(p)$ for $p \in A$.

Now order $\mathscr{F}$ the set of all finite subsets of A by inclusion. Then $F' \supset F$ implies $f_{F'}(x) \le f_F(x)$ for all x, so $\operatorname{Lim}_{F \in \mathscr{F}} f_F(x) = f(x)$ exists for every x, and $f \in B^1_M(H, R)$ by Proposition 3. Also $f(p) = \operatorname{Lim}_{F \in \mathscr{F}} f_F(p) = \operatorname{Lim}_{F \in \mathscr{F}, p \in F} f_F(x) = f_0(p)$ for $p \in A$, and $Df(p)\cdot z = \operatorname{Lim}_{F \in \mathscr{F}, p \in F} Df_F(p)\cdot z = f_1(p)\cdot z$ for all z in H and p in A, so $Df(p) = f_1(p)$. $f_F(x) \ge \inf_{y \in A} d_y(x)$ for all F gives $f(x) \ge \inf_{y \in A} d_y(x)$. Finally, $g \in B^1_M(H, R)$, $g(p) = f_0(p)$, and $Dg(p) = f_1(p)$ for $p \in A$ implies $g(x) \le f_F(x)$ for all F, so $g(x) \le f(x)$.

Corollary 1. *Let A be a closed subset of a Hilbert space H. Then there is an $f \in B^1_M(H, R)$ with $f(x) \ge \frac{1}{4}Md^2(x, A)$, and $g(x) \le f(x)$ if $g \in B^1_M(H, R)$ and $g(A) = Dg(A) = 0$.*

Proof. Take $f_0 = f_1 = 0$ on A. Then $d_y(x) = \frac{1}{4}M(y - x)^2$, and the corollary follows.

Remark. If A is convex, then $\frac{1}{2}Md^2(x, A) \in B^1_M(H, R)$ by Proposition 7, and $f(x) \le \frac{1}{2}Md^2(x, A)$ by Proposition 1. So $f(x) = \frac{1}{2}Md^2(x, A)$.

Corollary 2. *Any locally finite open cover $\{V_i\}$ of a Hilbert space H is the supporting set for a C^1 partition of unity.*

Proof. Find $f_i \in B^1_4(H, R)$ with $f_i(x) > d^2(x, H - V_i)$. Then $V_i = f_i^{-1}(R^+)$, and by defining $\varphi_i(x) = f_i(x)/\sum_j f_j(x)$ we have a C^1 partition $\{\varphi_i\}$ of unity with $V_i = \varphi_i^{-1}(R^+)$. Actually $\varphi_i \in U^1(H, R)$ in the sense of the remark following Corollary 2 of § 2.

Corollary 3. *$C^1(H, F)$ is uniformly dense in $C^0(H, F)$ for a Hilbert space H and any Banach space F.*

Corollary 4. *Given A and B closed in a Hilbert space H with $d(A, B) = \delta > 0$, there is an $f \in B^1_{4/\delta^2}(H, R)$ with $0 \le f(x) \le 1$ and $f(A) = 0$ and $f(B) = 1$.*

Proof. Let $B' = \{x \mid d(x, A) \ge \delta\}$. Let $f_0(A) = 0$, $f_0(B') = 1$, $f_1(A) = f_1(B^1) = 0$. Then (5) holds with $M = 4/\delta^2$, and we have $f \in {}^1_{4/\delta^2}(H, R)$ with $f(A) = 0$, $f(B') = 1$.

Since $d(x, (A \cup B')) \le \delta$ for all x, $m = \sup f(x) < \infty$. Suppose $m > 1$, and find a sequence x_n in $H - B'$ with $f(x_n) \to m$ and a sequence $z_n \in A$ with

$\|x_n - z_n\| < \delta$. Then $m \geq f(x_n + \delta(\frac{1}{4}Df(x_n))) \geq f(x_n) + \frac{1}{8}\delta^2\|Df(x_n)\|^2$ by Proposition 1. So $\|Df(x_n)\| \to 0$. But then (5) implies $m = \operatorname{Lim}_n |f(x_n) - f(z_n)| \leq 1$, a contradiction, so $m \leq 1$ and $0 \leq f(x) \leq 1$.

Corollary 5. *Suppose A is closed in Hilbert space H, and $f_0: H \to R^n$ and $f_1: H \to L(H, R^n)$ with*

$$\langle u, f(y)\rangle \leq \langle u, f(x)\rangle + \langle u, \tfrac{1}{2}(Df(x) + Df(y))[y - x]\rangle + \tfrac{1}{4}M(x-y)^2 - \tfrac{1}{4}(\langle u, Df(y) - Df(x)\rangle)^2/M$$

for all $x, y \in H$ and $u \in R^{n*}$, $\|u\| = 1$. Then there is an $f \in B^1_{M\sqrt{n}}(H, R^n)$ such that $f(x) = f_0(x)$ and $Df(x) = f_1(x)$ for x in A.

Proof. Let $e_1, \cdots, e_n$ be an orthonormal basis for R^n, extend $\langle f_0, e_i\rangle$ to $f^1, \cdots, f^n$ and set $f(x) = f^1(x)e_1 + \cdots + f^n(x)e_n$.

Corollary 6. *Given $g(x) \in B^0_M(H, R)$, a Hilbert space H and an $\varepsilon > 0$, there is an $f \in B^1_{M^2/\varepsilon}(H, R)$ with $|f(x) - g(x)| < \varepsilon$ for all x.*

Proof. Let $A_n = g^{-1}(n\varepsilon)$, $n = 0, \pm 1, \pm 2, \cdots$. Then $d(A_n, A_{n+1}) \geq \varepsilon/M$, and by Corollary 4 we can find $f_n \in B^1_{M^2/\varepsilon}(H, R)$ with $f_n(A_n) = n\varepsilon$, $f_n(A_{n+1}) = (n+1)\varepsilon$ and $n\varepsilon \leq f_n \leq (n+1)\varepsilon$. Let $f(x) = n\varepsilon$ if $x \in A_n$, and $f(x) = f_n(x)$ if $n\varepsilon < f(x) < (n+1)\varepsilon$.

Remark. This corollary is not true if R is replaced by l^2. Take $H = l^2$, and let $\sigma(x) = \sum_i |x_i| e_i$ where $\{e_i\}$ is an orthonormal basis. Then $\sigma \in B^0_1(l^2, l^2)$, but $\sup_{\|x\| \leq 1} \|f(x) - \sigma(x)\| \geq 1$ for $f \in B^1(l^2, l^2)$. This was proved in Wells [12].

5. B^2 functions and some open problems

The corollary of the next theorem shows that Corollary 4 of § 4 is not true if B^1 is replaced by B^2 even for A convex and bounded.

Theorem 1. *Suppose $f \in B^2_M(R^N, R)$, $f(A) = 0$, and $f(x) \geq 1$ when $d(x, A) \geq 1$ where $A = \{x \mid x_i$ (i-th coordinate of x) $\leq 0, \|x\| \leq 1\}$. Then $N < M^2 + 36M^4$.*

Proof. Assume $f \in B^2_M(R^n, R)$, $f(A) = 0$, $f(\{x \mid d(x, A) \geq 1\}) \geq 1$ and $N \geq M^2 + 36M^4$. Let $g(x) = \sum_{p \in S_N} f(p(x))/N!$ where S_N is the set of all permutations of the N coordinates of x. Then $g \in B^2_M(R^n, R)$ with $g(A) = 0$ and $g(\{x \mid d(x, A) \geq 1\}) \geq 1$. Define points y^n for $n = 0, \cdots, M^2$ with $y^n_i = 1/M$ for $i = 1, \cdots, n$, $y^n_i = -1/M$ for $i = n+1, \cdots, M^2$, and $y^n_i = 0$ for $i = M^2 + 1, \cdots, N$. Define z^n for $n = 1, \cdots, M^2$ with $z^n_i = 1/M$ for $i = 1, \cdots, n-1$, $z^n_n = 0$, $z^n_i = -1/M$ for $i = n+1, \cdots, M^2$, and $z^n_i = 0$ for $i = M^2 + 1, \cdots, N$. By symmetry, $\dfrac{\partial g}{\partial x_n}(z^n) = \dfrac{\partial g}{\partial x_m}(z^n)$ for $m = M^2 + 1, \cdots, N$. So

$$\left|\frac{\partial g}{\partial x_n}(z^n)\right|^2 \leq \frac{1}{36M^4} \sum_{m=M^2+1}^{N} \left|\frac{\partial g}{\partial x_m}(z^n)\right| \leq \frac{1}{36M^4}\|Dg(z^n)\|^2 \leq \frac{1}{36M^2},$$

or $\left|\frac{\partial g}{\partial x_n}(z^n)\right| \leq \frac{1}{6M}$. Now by Proposition 1,

$$g(y^n) \leq g(z^n) + \frac{1}{M}\frac{\partial g}{\partial x_n}(z^n) + \frac{1}{2}\frac{1}{M^2}\frac{\partial^2 g}{\partial x_n^2}(z_n) + \frac{M}{6}\left(\frac{1}{M}\right)^3$$
$$\leq g(z^n) + \frac{1}{6M^2} + \frac{1}{2}\frac{1}{M^2}\frac{\partial^2 g^n}{\partial x_n^2}(z^n) + \frac{1}{6M^2},$$

$$g(y^{n-1}) \geq g(z^n) - \frac{1}{6M^2} + \frac{1}{2}\frac{\partial^2 g}{\partial x_n^2}(z_n) - \frac{1}{6M^2},$$

so $g(y^n) \leq g(y^{n-1}) + \frac{2}{3}M^{-2}$. Summing up from $n = 1, \cdots, M^2$ gives $g(y^{m^2}) \leq g(y^0) + 2/3$. But $y^0 \in A$ with $g(y^0) = 0$, and $d(y^{M^2}, A) = 1$ with $g(y^{M^2}) \geq 1$, a contradiction. Hence $N < M^2 + 36M^4$.

Corollary 1. *Let $A = \{x | x \in l^2, x_i \leq 0, \|x\| \leq 1\}$, and suppose $f \in C^2(l^2, R)$ with $f(A) = 0$ and $f(\{x | d(x, A) \geq 1\}) \geq 1$. Then $f \notin B^2(l^2, B)$.*

Proof. Obvious from the theorem.

Corollary 2. *There exist a closed subset of l^2 and functions $f_0, f_1, f_2, f_3 : A \to R, L(l^2, R), L_s^2(l^2, R), L_s^3(l^2, R)$ satisfying the conditions of the Whitney extension theorem with the property that there is no C^3 or B^2 function agreeing with f_0 on the closed set.*

Proof. Let $A = \{x | x_1 = 1, x_i \leq 0$ for $j = 2, 3, \cdots$, and $\|x - e_1\| \leq 1\}$, and $B = \{x | x_1 = 1, d(x, A) \geq 1\}$. Let CA and CB be the cones formed on A and B with the origin. Define $f_0(x) = x_1^8$, $f_1(x)[h] = 8x_1h_1$, $f_2(x)[h] = 56x_1^6h_1^2$, $f_3(x)[h] = 336x_1^5h_1^3$ for $x \in CA$, and $f_0(x) = f_1(x) = f_2(x) = f_3(x) = 0$ on CB. Then it is easy to see that these functions satisfy the hypotheses of the Whitney extension theorem. If $f \in C^3(l^2, R)$ or $B^2(l^2, R)$, and $f|_{CA \cup CB} = f_0(x)$, then in the first case $D^3f(x)$ is bounded near zero, and in either case $f|_{x_1 = a} \in B^2(\{x | x_1 = a\}, R)$ for some $a > 0$. But this is impossible by Corollary 1. q.e.d.

We list some open problems:

(1) Does $\|x\| \in C^1(E - \{0\}, R)$ imply $d(x, A) \in C^1(E - A, R)$ whenever A is convex and closed?

(2) Do nonseparable $\mathscr{L}^p, p > 2$, have C^1 partitions of unity?

(3) Does nonseparable Hilbert space have C^2 partitions of unity?

(4) Is Theorem 2 of § 4 true for Banach-valued functions on H or for functions on non-Hilbertian Banach spaces with an appropriate change in (1)?

Added in proof. Since the submission of this paper Henryk Taruńcyk has obtained in [9] results which settle questions 2 and 3. We summarize some of these results:

(i) A Banach space E admits C^p, $p = 1, 2, \cdots; \infty$ partitions of unity if and only if there are a set A and a homeomorphic imbedding $u: E \to c_0(A)$ with $p_\alpha \circ u(x) \in C^p$ for all $\alpha \in A$ where p_α is the projection of $c_0(A)$ on its α-th coordinate.

Thus Taruńcyk observes that any Hilbert space $\ell_2(B)$ has C^∞ partitions of unity by taking $A = B \cup \{1\}$ and by defining $u(x)$ by

$$p_\alpha \circ u(x) = \|x\|^2 \qquad \text{for } \alpha = 1$$
$$= x_\beta \qquad \text{for } \alpha = \beta,\ \beta \in B\,.$$

(ii) If E is a reflexive Banach space with an equivalent locally uniformly convex norm of class C^p, then E admits C^p partitions of unity.

Thus $\mathscr{L}^p$ has C^∞ p.o.u. if p is an even integer, and C^{p-1} p.o.u. if p is an odd integer.

(iii) A Banach space E has C^p p.o.u. if and only if there is a σ-locally finite base of the topology of E consisting on nonzero sets of real valued functions of class C^p.

(iv) In a personal communication Taruńcyk has shown that E has B^p p.o.u. $p < \infty$ if there is a σ-discrete base of the topology of E consisting of nonzero subsets of real valued functions of class B^p. The author has proved the converse statement.

This generalizes Theorem 1. Also using Corollary 1 and the fact that every metric space has a σ-discrete base for the topology, it follows that every Hilbert space admits B^1 partitions of unity.

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